Dynamic Behavior and Topological Structure of the Universe with Dark Energy

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The topology of the space-time has intimate relationship with the dynamical behavior of the Universe. Via qualitatively analyzing the relationship, this letter shows that, the noncyclic, flat and hyperbolic cosmological models might lead to some nonphysical effects. Therefore only the cyclic and compact cosmological model with a tiny or vanishing cosmological constant is most natural and acceptable in physics. Obviously, if such constraints can be verified by observation, they would be much helpful for the researches on the properties of dark matter and dark energy in cosmology.

Keywords: *cosmic curvature, cosmological constant, negative pressure, dark matter, dark energy*

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The most fundamental problems in cosmology are the determination of cosmic curvature K, cosmological constant Λ and the equation of state of dark matter and dark energy. They are intimately coupled with each other by the Friedmann equation and energy conservation law, so that one can hardly determine them separately. Some characteristic parameters of the universe such as the age $\mathcal{T}$, the Hubble's constant H_0, the total mass density $\Omega_{\rm tot}$ and so on have been measured to high accuracy[1, 2, 3, 4]. Since the Friedmann equation is a dynamical equation, the properties of the solution can not be statically determined by these parameters.

For the cosmic curvature K, the usual method is to transform the Friedmann equation into a static equation $\Omega_K \equiv K\dot{a}^{-2} = \Omega_{\rm tot} - 1$, then the case $K = 0, \pm 1$ can be judged by examining the empirical data $\Omega_{\rm tot} > 1, = 1$ or < 1. However, as computed in [5], we always have $\Omega_{\rm tot} \approx 1$ for a young universe no matter what case of the spatial curvature is. To

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examine the behavior of a dynamical equation according to the static algebraic equation is usually unreliable, so we can hardly use this ambiguous criterium to determine K.

The cosmological constant Λ has a checkered history. Since Einstein introduced Λ in 1917 to get a static and closed universe, whether $\Lambda = 0$ or not has been debated many times[6]-[9]. However, this is just a matter of value, which can be determined by empirical data and explained by the harmonious relationship between the cosmological variables. The situation of the cosmic curvature K is quite different, which represents the essentially distinguishing topology of the universe. Some authors suggested to use the time dependent $\Lambda(t)$. This means to treat Λ as a scalar field, and then this model is actually equivalent to the so called quintessence or phantom models[6].

The dark matter and dark energy attract the attention of the scientists all over the world and become the hottest topic. They challenge the traditional standard models of particles and cosmology, but their nature still hide in the mysterious darkness. The usual description of dark matter and dark energy is using the equation of state $P = w\rho$ together with $w = w(a)$ or $w = w(z)$. There are many specific models to fit the observational data, which can be obtained from review papers like [6]-[15]. However, the resolutions to the problem are far beyond clearness and include anomalies[16].

The purpose of this letter is to constraint K and Λ via qualitative analysis for the dynamical behavior of the Friedmann equation, where only some general properties of the dark matter or energy are involved, and these properties are abstracted from many established models [12, 13, 14, 15]. Some similar discussions with concrete gravitating sources were once performed by many authors[17]-[24]. In [17, 18, 19], the nonlinear scalar filed is discussed, and the cyclic universe is obtained. In [20], a number of exact cyclic solutions with normal dust and radiation were obtained, and the exact solution with a ghost field and electromagnetic field was derived in [21]. The quantized nonlinear spinor model and the trajectories were calculated in [22, 23]. The Friedmann equation for some well-known dark energy models were translated as the dynamics of Hamiltonian system by introducing a potential function $V(a)$, and the evolution trajectories are analyzed in [13].

In this letter, under some general assumptions upon the mass density and pressure, we found that some definite constraints on the topology of the universe and the cosmological constant Λ can be derived. These treatments show the power of logic in the physical study, and the results would be much helpful for the researches on dark matter and dark energy as well as the other issues in cosmology.

In average sense, the universe is highly isotropic and homogeneous, and the metric is described by Friedmann-Robertson-Walker(FRW) metric. The corresponding line element is usually given by

$$ds^2 = d\tau^2 - a^2(\tau)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{1}$$

where $K = 1,\ 0$ and -1 correspond to the closed, flat and open universe respectively. However, in this form, the solution $a(t)$ can not be expressed by elementary functions, and is nonanalytic as $a \to 0$ (e. g. $a \propto t^{\frac{1}{2}}$ or $a \propto t^{\frac{2}{3}}$). Secondly, the Friedmann equation

$$\dot{a}^2 = -K + \frac{1}{3}\Lambda a^2 + \frac{8\pi G}{3}\rho_{\rm m} a^2, \quad \left(\dot{a} = \frac{da}{d\tau}\right) \tag{2}$$

includes singular term $\rho_{\rm m} a^2 \to \infty$ as $a \to 0$, which increases difficulties for discussion[13]. In this letter, we use the conformal coordinate system for the convenience of analysis. The line element becomes

$$ds^2 = a(t)^2\left(dt^2 - dr^2 - \mathcal{S}(r)^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{3}$$

where $dt = a^{-1}d\tau$ is the conformal time,

$$\mathcal{S} = \begin{cases} \sin r & \text{if} \quad K = 1, \\ r & \text{if} \quad K = 0, \\ \sinh r & \text{if} \quad K = -1. \end{cases} \tag{4}$$

Then the Friedmann equation (2) becomes

$$a'^2 = -Ka^2 + \frac{1}{3}\Lambda a^4 + \frac{8\pi G}{3}\rho_{\rm m} a^4, \tag{5}$$

where $\rho_{\rm m}$ is the total mass-energy density of all gravitating sources except for the geometrical components K and Λ, but including particles, radiation, dark matter, dark energy and so on. The mass-energy density $\rho_{\rm m}$ satisfies the energy conservation law

$$\frac{d}{da}(\rho_{\rm m} a^3) = -3Pa^2. \tag{6}$$

In cosmology, although we call P pressure, but it is actually a variable not only including the usual pressure of thermal movement of micro particles, but also including the potential energy of all fields[12, 22, 23]. So $P < 0$ is possible in physics.

Now we examine some general characteristics of $\rho_{\rm m}$ and P as functions of a. By their physical meaning and the observational facts, we have two conclusions:

A1. The total mass-energy density is always positive, namely

$$\rho_{\rm m} > 0. \tag{7}$$

A2. The pressure $P < 0$ when the Universe is small or equivalently $a \to 0$. P is finite if the spacetime is nonsingular, that is, for any given tiny positive number $\delta > 0$, there exists a finite number $P_\delta < \infty$, such that we have

$$|P(a)| \le P_\delta, \qquad (\forall a \ge \delta). \tag{8}$$

(7) is a result of positive definite Hamiltonian of matter, and $P < 0$ is an observational fact. For the nonlinear spinors[25] and most famous dark energy models[12], (A1) and (A2) hold. (A1) and (A2) are the basic assumptions for the following discussion. Denoting the comoving mass-energy density $M = \rho_{\rm m} a^3$, by (6), (8) and the Gronwall inequality, we have

$$\left|\frac{dM}{da}\right| \le 3P_\delta a^2,\ M \le R_0 + P_\delta a^3,\ \rho_{\rm m} \le \frac{R_0}{a^3} + P_\delta,\ (\forall a \ge \delta) \tag{9}$$

where R_0 is a constant. This means that, under the constraint of the energy conservation law, $M(a)$ and $\rho_{\rm m}(a)$ are continuous functions of a with bounded first order derivatives, despite the explosive transition of matter occurs in the universe.

In [12], the authors ranked 10 dark energy models and 10 modified Friedmann equations according to the Bayesian information criteria. In this letter, we are not concerned for the effectiveness of these models, but concerned for the common features of their Friedmann equation. The equation generally takes the following form

$$H^2 = H_0^2\left(-\Omega_f(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda + \Omega_X(z)\right), \tag{10}$$

where

$$H = \frac{da}{ad\tau} = \frac{a'(t)}{a^2} \tag{11}$$

is the Hubble's parameter, $H_0 = 70 \pm 4{\rm km\ s^{-1}Mpc^{-1}}$ is the present value of H, $z = \frac{a(t_a)}{a(t)} - 1$ is the redshift and t_a the present time, $(\Omega_m, \Omega_K, \Omega_\Lambda)$ are the dimensionless energy densities corresponding to common matter, curvature and cosmological constant respectively. $\Omega_f > 0$ is caused by potential of fields such as spinors [22, 23] and the Casimir effect of massless scalar[12], $\Omega_X(z) > 0$ stands for the energy density of other dark matter and dark energy as well as the effects of modification of general relativity, which is different from the previous terms. Since it is inadequate to analysis a dynamical equation (10) as algebraic equation,

we translate it into the dynamical from. More generally, the detailed Friedmann equation should take the following form

$$a'^2 = F(a), \quad F \equiv -\rho_f + \frac{\rho_T b}{a+\sqrt{a^2+b^2}} + 2Ra - Ka^2 + \frac{1}{3}\Lambda a^4 + X(a), \tag{12}$$

where $\rho_f > 0$ corresponds to negative potential of fields, $\rho_T > 0$ to the kinetic energy of particles[26], R to the total comoving mass-energy density including mass-energy density of dark matter and dark energy, $X(a)$ correspond to the unknown parts of the dark matter and dark energy, which is different from the previous terms and usually take small values.

For ordinary matter, it is equivalent to set

$$\rho_T = X(a) = 0, \qquad \rho_f = -C_{\rm r}, \qquad R = \frac{1}{2}C_{\rm m} \tag{13}$$

in equation (12), where $C_{\rm r}$ is a constant representing the strength of energy density of radiation, and $C_{\rm m}$ a constant representing the strength of mass-energy density of massive particles. In this case, a detailed analysis on the evolving behaviors of the solution with different coefficients were given in [20]. Since $C_{\rm r} \ge 0$, which leads to positive pressure if $a \to 0$, and then the initial singularity is inevitable. Together with the initial condition $a(0) = a_0$, the dynamical equation (12) becomes a closed system, and then the solution is uniquely determined. So the introduction of the entropy, which is an ambiguous concept in cosmology, should be merged into the equation of state $P(\rho)$ and $X(a)$.

The auxiliary function $F(a)$ is also continuous function with bounded first order derivative for $a \ge 0$ according to (9). The concrete form of $X(a)$ is not important for the following discussion.

By (12) we find R is the mean scale of the universe, which can be used as length unit[22, 23]. Comparing (12) with (5), we get the mass-energy density in the usual sense

$$\rho_{\rm m} = \frac{3}{8\pi G a^4}\left(F(a) + Ka^2 - \frac{1}{3}\Lambda a^4\right), \tag{14}$$

$$= \frac{3}{8\pi G a^4}\left(-\rho_f + \frac{\rho_T b}{a+\sqrt{a^2+b^2}} + 2Ra + X(a)\right). \tag{15}$$

Substituting (15) into (6), we get the pressure

$$P = -\frac{1}{8\pi G a^4}\left(\rho_f - \frac{\rho_T b}{\sqrt{a^2+b^2}} + X'(a)a - X(a)\right). \tag{16}$$

In the case of $X(0) = X'(0) = 0$, by the condition (A2) $P < 0$, we find that $F(a) = 0$ has a positive root $a_0 \to 0$. If not, let $a \to 0$, by $P < 0$ we have

$$P \to -\frac{1}{8\pi G a^4}(\rho_f - \rho_T) < 0. \tag{17}$$

Consequently, by the definition of $F(a)$ in (12), we get

$$F(0) = -\rho_f + \rho_T < 0. \tag{18}$$

On the other hand, for the practical solution of Friedmann equation, we should have $F(a) = a'^2 \geq 0$. By (18) and the continuity of $F(a)$, the equation $F(a) = 0$ should have one positive root a_0. This is a contradiction, so we have $0 < a_0 \approx \frac{\rho_f - \rho_T}{2R} \ll R$.

In the case of $X \to X_0 a^{-n}, (a \to 0, n > 0)$, we have

$$P \to \frac{(n+1)X_0}{8\pi G a^{4+n}}, \qquad (a \to +0). \tag{19}$$

by $P < 0$ we find $X_0 < 0$. According to the definition of $F(a)$ in (12), we get

$$F \to \frac{X_0}{a^n} < 0, \qquad (a \to +0). \tag{20}$$

Combining it with $F(a) = a'^2 \geq 0$ and the continuity of $F(a)$, we find $F(a) = 0$ has a positive root a_0. This is a contradiction. So in the history of the Universe, $F(a) = 0$ certainly has a positive root $a = a_0 \to +0$.

If $F(a) = 0$ only has this positive real root a_0, then $F(a)$ can be expressed as

$$F(a) = (a - a_0)A(a), \quad (A > 0,\ \forall a \geq a_0). \tag{21}$$

If $F(a) = 0$ has multi different positive real roots $0 < a_0 < a_1 < a_2 < \cdots$, then $F(a)$ can be expressed as

$$F(a) = (a - a_0)(a_1 - a)B(a), \quad (B > 0,\ a_0 \leq a \leq a_1). \tag{22}$$

The connected phase trajectories $a \sim a'$ of dynamical equation (12) with (21) or (22) are displayed in FIG.1, in which we have set the mean scale $R = 1$. (22) corresponds to the cyclic cosmological model, and (21) to the noncyclic one. We set the time origin $t = 0$ at the turning point $a(0) = a_0$.

In (21) and (22), $a_0 > 0$ is derived from $P < 0$ while $a \to 0$. As a matter of fact, $a_0 < 0$ also leads to other nonphysical effects. Except for the violation of (A1) $\rho_{\rm m} > 0$, the corresponding $a \sim a'$ trajectories with $a_0 < 0$ must cross the singularity $a(t) = 0$ as displayed in FIG.2. Such phenomenon is obviously nonphysical. In the case $a_0 = 0$, which corresponds to the Big Bang model, it is unstable for the perturbation of parameters. Noticing the FRW metric holds in mean sense, a reliable solution in physics should be stable under small perturbation of parameters.

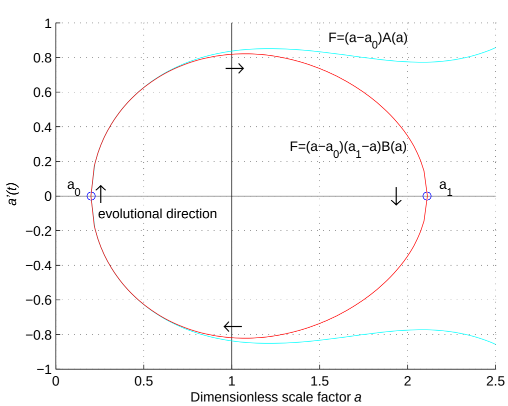

Figure 1: The regular trajectories

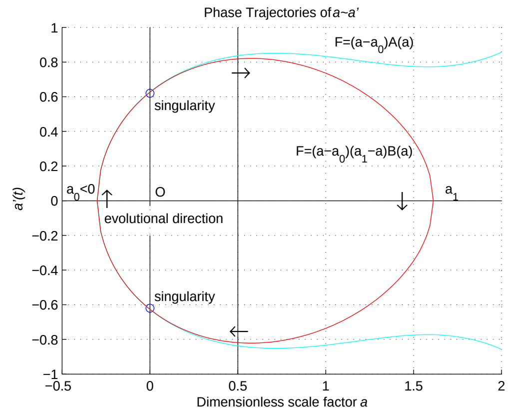

Figure 2: Singularities occur if $a_0 < 0$

There may be the case of multiple roots, whose phase trajectory has the form similar to ∞. However this case is unstable under perturbation, so it is only meaningful in mathematics rather than in physics.

Substituting (21) or (22) into (14), we get a decisive criterion for K. When $a \longrightarrow a_0$, for both cases, we have

$$\rho_{\rm m}(a_0) = \frac{3}{8\pi G a_0^2}\left(K - \frac{1}{3}\Lambda a_0^2\right). \tag{23}$$

If $\Lambda \geq 0$ as usually realized or $|\Lambda a_0^2| \ll 1$, we find $K = 1$ by $\rho_{\rm m} > 0$, namely, the space should be closed sphere S^3.

If $\Lambda < 0$, we show that the noncyclic model (21) include nonphysical effect or logical contradiction. We examine the asymptotic behavior of $F(a)$ in (12) as $a \to \infty$. If $X(a)a^{-4} \to 0$, then $F(a) \to \frac{1}{3}\Lambda a^4 < 0$, which contradicts (21). If there exists numbers $(A_1 > 0, C > 0)$, such that $X(a) \geq C a^n$, $(n > 4)$ when $a \geq A_1$, then by the asymptotic behavior of $a(t)$, we find that $a \to \infty$ in finite past and finite future, which is nonphysical. Moreover P violates (8) and $\rho_{\rm m}$ violates (9). The rest case is something similar to $X(a) \to C a^4 \ln(a)$. For this case, we also have that P violates (8) and $\rho_{\rm m}$ violates (9). Therefore, $\Lambda < 0$ is inconsistent with the noncyclic model.

For cyclic closed case, we have an estimation of upper bound for the cosmological constant Λ. Substituting (22) into (14), we have

$$\rho_{\rm m}(a_1) = \frac{3}{8\pi G a_1^2}\left(1 - \frac{1}{3}\Lambda a_1^2\right) > 0. \tag{24}$$

Consequently, we get

$$\Lambda < \frac{3}{a_1^2} \approx \frac{3}{4R^2}. \tag{25}$$

The noncyclic model with closed space and positive Λ can not be ruled out by similar qualitative analysis. However such model might be inconsistent with the isotropy and homogeneity of the present Unverse, because the universe should be heavily anisotropy and inhomogeneity before the turning point $t < 0$ due to the lack of initial causality among remote parts, and some information should be kept today.

To sum up, by analyzing the dynamical behavior of the general Friedmann equation, we find that only the cyclic and closed cosmological model with a tiny or vanishing Λ is natural and reasonable in physics. These constraints would be greatly helpful for the research of some issues of cosmology.

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